Bayesian One Sample Prediction of Future GOS's From A Class of Finite Mixture Distributions Based On Generalized Type-I Hybrid Censoring Scheme

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Abstract: *In this paper, the Bayesian prediction intervals for a future gos's from a mixture of two components from a class of continuous distributions under generalized Type-I hybrid censoring scheme are computed. We consider the one sample prediction technique. A mixture of two Weibull components model is given as an application. Our results are specialized to upper order statistics and upper record values. The results obtained by using the Markov Chain Monte Carlo (MCMC) algorithm.*

Keywords: *Generalized order statistics; Bayesian prediction; One-sample scheme; Finite mixtures; Generalized Type-I hybrid censoring scheme; MCMC algorithm.*

Introduction

In many practical problems of statistics, one wishes to use the results of a previous data to predict the results of a future data from the same population. One way to do this is to construct an interval, which will contain these results with a specified probability. This interval is called the prediction interval. Prediction has been applied in medicine, engineering, business and other areas as well. For details on the history of statistical prediction, analysis and applications, see for example, Aitchison and Dunsmore [5], Geisser [24], Dunsmore [21], Howlader and Hossain [26], AL-Hussaini ([6], [7]), Corcuera and Giummolè [20], Nordman and Meeker [32], Ahmadi et al.[3], Ahmadi et al.[4], Ateya [13], Ahmad et al.[2], Balakrishnan and Shafay [14] and Shafay and Balakrishnan [34].

Several authors have predicted future order statistics and records from homogeneous and heterogeneous populations that can be represented by single-component distribution or finite mixtures of distributions, respectively. For more details, see AL-Hussaini et al.[11], AL-Hussaini and Ahmad ([9], [10]), Ali Mousa and AL-Sagheer [12] and AL-Hussaini [8].

The two most popular censoring schemes are Type-I and Type-II censoring schemes. The hybrid censoring scheme is the mixture of Type-I and Type-II censoring schemes. It was introduced by Epstein [23].

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In hybrid censoring scheme the life-testing experiment is terminated at a random time $T_1^* = \min\{X_{r:n}, T\}$, where $r \in 1,2,\ldots,n$ and $T \in (0,\infty)$ are fixed in advance. Following Childs et al. [19], we will refer to this scheme as Type-I hybrid censoring scheme (Type-I HCS), since under this scheme the time on test will be no more than T. Recently, it becomes quite popular in the reliability and life-testing experiments, see for example, the work of Chen and Bhattacharya [18], Gupta and Kundu [25], Kundu [30] and Kundu and Howlader [31].

Noting that this scheme, which would guarantee the experiment to terminate by a fixed time T, may result in few failures, for this reason, Childs et al.[19] proposed a new HCS, referred to as Type-II hybrid censoring scheme (Type-II HCS), which guarantees a fixed number of failures. Inference based on Type-II hybrid censored data from Weibull distribution by Banerjee and Kundu [15]. Though the Type-II HCS guarantees a specified number of failures, it has the disadvantage that it might take a very long time to observe r failures and complete the life test.

Chandrasekar, et al.[17] found that both Type-I and Type-II HCS's have some potential drawbacks. Specifically, in Type-I HCS, there may be very few or even no failures observed whereas in Type-II HCS the experiment could last for a very long period of time. So, they suggest generalized hybrid censoring schemes.

Generalized order statistics ($gos's$) concept was introduced by Kamps [28] as a unified approach to several models of ordered random variables such as upper order statistics ($uos's$), upper record values ($urv's$), sequential order statistics, ordering via truncated distributions and censoring schemes, see for example, Kamps and Gather [29], AL-Hussaini [8], Jaheen [27] and Ahmad [1].

Let us consider a general class of continuous distributions that suggested by AL-Hussaini and Ahmad ([9], [10]) with cumulative distribution function (CDF) $F(x)$ given by $F(x) = F(x;\theta) = 1 - \exp[-\alpha\lambda_\beta(x)], \quad x \ge 0, (\alpha, \beta > 0), \quad (1.1)$ where $\theta = (\alpha, \beta)$ and $\lambda_\beta(x) = \lambda(x;\beta)$, is non-negative, continuous, monotone increasing and differentiable function of x such that $\lambda(x;\beta) \to 0$ as $x \to 0^+$ and $\lambda(x;\beta) \to \infty$ as $x \to \infty$. The probability density function (PDF) of this class is given by $f(x) = \alpha\lambda'_\beta(x)\exp[-\alpha\lambda_\beta(x)], \quad x \ge 0.$

This class of absolutely continuous distributions including, as special cases, Weibull (exponential, Rayleigh as special cases), compound Weibull (or three parameters Burr-type XII), Pareto, power function (beta as a special case), Gompertz and compound Gompertz distributions, among others.

The corresponding reliability function (RF) and the hazard rate function (HRF) are given, respectively by

$R(x) = \exp[-\alpha\lambda_\beta(x)], \quad x \geq 0,$

$h(x) = \alpha\lambda'_\beta(x), \quad x \geq 0.$ The CDF of finite mixture of two components $F_1(x)$ and $F_2(x)$ from a class (1.1), is given, for $0 \leq p_1 \leq 1$, by $F(x) = p_1F_1(x) + p_2F_2(x)$, where $p_1 = p,\ p_2 = 1 - p_1$. For $i = 1,2,\ F_i(x)$ from (1.1) is, $F_i(x) = 1 - \exp[-\alpha_i\lambda_{\beta_i}(x)], \quad x \geq 0.$

The PDF of finite mixture $f(x)$ is given by $f(x) = p_1 f_1(x) + p_2 f_2(x),$ (1.2)
where, for $i = 1,2,\ \ f_i(x) = \alpha_i\lambda'_{\beta_i}(x)\exp[-\alpha_i\lambda_{\beta_i}(x)], \quad x > 0,$ hence the CDF of a finite mixture $F(x)$ of such two components is $F(x) = 1 - p_1\exp[-\alpha_1\lambda_{\beta_1}(x)] - p_2\exp[-\alpha_2\lambda_{\beta_2}(x)].$

The corresponding RF and HRF are given, respectively, by $R(x) = p_1R_1(x) + p_2R_2(x),$ (1.3) $H(x) = f(x)/R(x).$ For the generalized Type-I HCS, the likelihood function can be written, see Chandrasekar et al.[17], when $m \neq -1$ and $m = -1$, respectively, as

$$L(\theta\,|\,x) = \begin{cases} c_{D_1-1}[R(T_1)]^{\gamma_{D_1}-1}\displaystyle\prod_{i=1}^{D_1}[R(x_i)]^m f(x_i), & D_1 = r,\dots,n, \\ c_{r-1}[R(x_r)]^{\gamma_{r+1}}\displaystyle\prod_{i=1}^{r}[R(x_i)]^m f(x_i), & D_1 = 0,1,\dots,r-1; D_2 = r, \\ c_{D_2-1}[R(T_2)]^{\gamma_{D_2}-1}\displaystyle\prod_{i=1}^{D_2}[R(x_i)]^m f(x_i), & D_2 = 0,\dots,r-1, \end{cases} \tag{1.4a}$$

$$L(\theta\,|\,x) = \begin{cases} k^{D_1}[R(T_1)]^{k-1}\displaystyle\prod_{i=1}^{D_1}H(x_i), & D_1 = r,\dots,n, \\ k^r[R(x_r)]^{k-1}\displaystyle\prod_{i=1}^{r}H(x_i), & D_1 = 0,1,\dots,r-1; D_2 = r, \\ k^{D_2}[R(T_2)]^{k-1}\displaystyle\prod_{i=1}^{D_2}H(x_i), & D_2 = 0,\dots,r-1, \end{cases} \tag{1.4b}$$

where $x = (x_1,\dots,x_r)$, and $C_{t-1} = \prod_{j=1}^{t}\gamma_j,\ \ \gamma_t = k + (n-t)(m+1).$

We shall use the conjugate prior density, that was suggested by AL-Hussaini ([6], [7]), in the following form

$$\pi(\theta;\nu) \propto C(\theta;\nu)\exp[-D(\theta;\nu)],\ \theta = (p,\alpha_1,\alpha_2,\beta_1,\beta_2),\ \nu \in \Omega, \tag{1.5}$$

where Ω is the hyper-parameter space. It follows, from (1.4a), (1.4b) and (1.5), that the posterior density function is given by

$$\pi^*(\theta\,|\,x) = A_1C(\theta;\nu)\exp[-D(\theta;\nu)]L(\theta\,|\,x), \tag{1.6}$$

where $A_1^{-1} = \int_\theta \pi(\theta;\nu)L(\theta\,|\,x)\,d\theta.$ In this paper, Bayesian prediction intervals ($BPI's$) for a future gos's are constructed when the previous (informative) sample is a finite mixture of two components from a general class of continuous distributions under generalized Type-I HCS. One-sample scheme is used in prediction. In Section 3, illustrative example of finite mixture of two Weibull components is discussed. Specialization is made in $uos's$ and $urv's$ cases. Conclusion remarks are presented in Section 4.

Bayesian One Sample Prediction Using MCMC Technique

Suppose that the first r gos's $X_{1;n,m,k}, X_{2;n,m,k}, \dots, X_{r;n,m,k}, 1 \le r \le n,$ have been formed and we wish to predict the future gos's $X_{r+1;n,m,k}, X_{r+2;n,m,k}, \dots, X_{n;n,m,k}$.

Let $X_a^* \equiv X_{r+a;n,m,k}$, $a = 1,2,\dots,n-r,$ the conditional PDF of the a^{th} future gos given the past observations X, can be written, see AL-Hussaini and Ahmad [9], as

$$h(x_a^* \mid \theta, x) \propto \begin{cases} \displaystyle\sum_{i=0}^{a-1} \omega_i^{(a)} [R(x_a^*)]^{\gamma_{r+a-i}-1} [R(x_r)]^{-\gamma_{r+a-i}} f(x_a^*), & m \ne -1, \\ \displaystyle\sum_{i=0}^{a-1} \omega_i^{(a)} [\ln R(x_a^*)]^i [\ln R(x_r)]^{a-i-1} [R(x_a^*)]^{k-1} & \\ \times [R(x_r)]^{-k} f(x_a^*), & m = -1, \end{cases} \tag{2.1}$$

where $\omega_i^{(a)} = (-1)^i \binom{a-1}{i}$.

Substituting (1.2) and (1.3) in (2.1) we have the two cases:

For $m \ne -1,$ the conditional PDF takes the form

$$h_1(x_a^* \mid \theta, x) \propto [p_1 f_1(x_a^*) + p_2 f_2(x_a^*)] \sum_{i=0}^{a-1} \omega_i^{(a)} [p_1 R_1(x_a^*) + p_2 R_2(x_a^*)]^{\gamma_{r+a-i}-1} \times [p_1 R_1(x_r) + p_2 R_2(x_r)]^{-\gamma_{r+a-i}}. \tag{2.2}$$

For $m = -1,$ the conditional PDF takes the form

$$h_2(x_a^* \mid \theta, x) \propto [p_1 f_1(x_a^*) + p_2 f_2(x_a^*)][p_1 R_1(x_a^*) + p_2 R_2(x_a^*)]^{k-1}$$
$$\times \sum_{i=0}^{a-1} \omega_i^{(a)} (\ln [p_1 R_1(x_a^*) + p_2 R_2(x_a^*)])^i (\ln [p_1 R_1(x_r) + p_2 R_2(x_r)])^{a-i-1} \times [p_1 R_1(x_r) + p_2 R_2(x_r)]^{-k}. \tag{2.3}$$

By multiplying (1.6) by (2.1) and then integrating with respect to $\theta \equiv p, \alpha_1, \alpha_2, \beta_1$ and β_2, the predictive PDF of $X_a^*, (a = 1,2,\dots,n-r)$ given the past observation x is given by

$$f^*(x_a^* \mid x) = \int_\theta h(x_a^* \mid \theta, x) \pi^*(\theta \mid x)\, d\theta, \quad x_a^* > x_r, \tag{2.4}$$

then the predictive survival function is given, for the a^{th} future gos's, by

$$P[X_a^* > \nu \mid x] = \int_\nu^\infty f^*(x_a^* \mid x) dx_a^*, \quad \nu > x_r. \tag{2.5}$$

A $100\,\tau\%$ BPI for X_a^* is then given by

$$P[L < X_a^* < U] = \tau,$$

where L and U are obtained, respectively, by solving the following two equations

$$P[X_a^* > L \mid x] = \frac{1+\tau}{2}, \tag{2.6}$$

$$P[X_a^* > U \mid x] = \frac{1-\tau}{2}. \tag{2.7}$$

Since the joint posterior density of the parameters $\pi^*(\theta \mid x)$ cannot be expressed in closed form and hence it cannot be evaluated analytically, so we propose to apply Metropolis algorithm to draw MCMC samples. Eberaly and Casella [22] were interested in the problem of estimating the posterior Bayesian credible region by the MCMC algorithm. Bayarri et al.[16] proposed MCMC algorithms to simulate from conditional predictive distributions.

This technique can be done by rewritten the predictive PDF (2.4), of X_a^*, given the past observations x, as

$$f^*(x_a^* \mid x) = \frac{\sum_{i=1}^{N} h(x_a^* \mid \theta_i, x)}{\sum_{i=1}^{N} \int_{x_r}^{\infty} h(x_a^* \mid \theta_i, x) dx_a^*}, \quad x_a^* > x_r, \tag{2.8}$$

where $\theta_i, i = 1,2,3,\ldots,N$ are generated from the posterior density function (1.6).

A $100\,\tau\%$ BPI (L,U) of the future observation X_a^* is given by solving the following two nonlinear equations

$$\frac{\sum_{i=1}^{N} \int_{L}^{\infty} h(x_a^* \mid \theta_i, x) dx_a^*}{\sum_{i=1}^{N} \int_{x_r}^{\infty} h(x_a^* \mid \theta_i, x) dx_a^*} = \frac{1+\tau}{2}, \tag{2.9}$$

$$\frac{\sum_{i=1}^{N} \int_{U}^{\infty} h(x_a^* \mid \theta_i, x) dx_a^*}{\sum_{i=1}^{N} \int_{x_r}^{\infty} h(x_a^* \mid \theta_i, x) dx_a^*} = \frac{1-\tau}{2}, \tag{2.10}$$

Numerical methods such as Newton-Raphson, are necessary to solve the above two equations to obtain L and U for a given τ.

Example (Two Weibull Components)

In this model, for $j = 1,2$ and $x > 0$, $\lambda_{\beta_j}(x) = x^{\beta_j}$, so $\lambda'_{\beta_j}(x) = \beta_j x^{\beta_j - 1}$.

Suppose that all parameters are unknown. Let p_1 be independent of α_1, α_2 and independent of β_1, β_2. As a suitable prior distribution of p, we consider the beta distribution with parameters b_1 and b_2 in the form $\pi(p_1) \propto p_1^{b_1 - 1} p_2^{b_2 - 1}$.

Suppose that α_1 and α_2 are distributed as gamma distributions with positive parameters (δ_1, d_1) and (δ_2, d_2), respectively, in the forms

$$\pi(\alpha_1) \propto \alpha_1^{\delta_1 - 1} \exp(-d_1 \alpha_1), \quad and \quad \pi(\alpha_2) \propto \alpha_2^{\delta_2 - 1} \exp(-d_2 \alpha_2),$$

and the prior distributions of β_1 and β_2 are gamma distributions with positive parameters (δ_3, d_3) and (δ_4, d_4), respectively, in the forms $\pi(\beta_1) \propto \beta_1^{\delta_3 - 1} \exp(-d_3\beta_1)$, *and* $\pi(\beta_2) \propto \beta_2^{\delta_4 - 1} \exp(-d_4\beta_2)$. Now, the joint prior density function of $\theta = (p_1, \alpha_1, \alpha_2, \beta_1, \beta_2)$ is given by

$$\pi(\theta) \propto p_1^{b_1-1} p_2^{b_2-1} \alpha_1^{\delta_1-1} \alpha_2^{\delta_2-1} \beta_1^{\delta_3-1} \beta_2^{\delta_4-1} \exp\ [-(d_1\alpha_1 + d_2\alpha_2 + d_3\beta_1 + d_4\beta_2)]. \tag{3.1}$$

Upper order statistics

In the *uos's* case from the case, $m \neq -1$ ($m = 0$ and $k = 1$), by multiplying the likelihood function (1.4a) and the prior density function (3.1), the joint posterior density function will be in the form

$$\pi^*(\theta \mid x) \propto \begin{cases} p_1^{b_1-1} p_2^{b_2-1} \alpha_1^{\delta_1-1} \alpha_2^{\delta_2-1} \beta_1^{\delta_3-1} \beta_2^{\delta_4-1} \\ \times \exp\ [-(d_1\alpha_1 + d_2\alpha_2 + d_3\beta_1 + d_4\beta_2)] \\ \times [p_1\xi_1(x_\ell) + p_2\xi_2(x_\ell)]^{(n-\ell)} \\ \times \prod_{i=1}^{\ell} [p_1\psi_1(x_i)\xi_1(x_i) + p_2\psi_2(x_i)\xi_2(x_i)], \ D = 0,1,\ldots,\ell-1, \\ \\ p_1^{b_1-1} p_2^{b_2-1} \alpha_1^{\delta_1-1} \alpha_2^{\delta_2-1} \beta_1^{\delta_3-1} \beta_2^{\delta_4-1} \\ \times \exp\ [-(d_1\alpha_1 + d_2\alpha_2 + d_3\beta_1 + d_4\beta_2)] \\ \times [p_1\xi_1(T) + p_2\xi_2(T)]^{(n-D)} \\ \times \prod_{i=1}^{D} [p_1\psi_1(x_i)\xi_1(x_i) + p_2\psi_2(x_i)\xi_2(x_i)], \ D = \ell,\ldots,r-1, \\ \\ p_1^{b_1-1} p_2^{b_2-1} \alpha_1^{\delta_1-1} \alpha_2^{\delta_2-1} \beta_1^{\delta_3-1} \beta_2^{\delta_4-1} \\ \times \exp\ [-(d_1\alpha_1 + d_2\alpha_2 + d_3\beta_1 + d_4\beta_2)] \\ \times [p_1\xi_1(x_r) + p_2\xi_2(x_r)]^{(n-r)} \\ \times \prod_{i=1}^{r} [p_1\psi_1(x_i)\xi_1(x_i) + p_2\psi_2(x_i)\xi_2(x_i)], \ D = r. \end{cases} \tag{3.2}$$

From (3.2), the posterior density of p_1 is

$$\pi^*(p_1 \mid \alpha_1, \alpha_2, \beta_1, \beta_2, x) \propto \begin{cases} p_1^{b_1-1} p_2^{b_2-1} [p_1\xi_1(x_\ell) + p_2\xi_2(x_\ell)]^{(n-\ell)} \\ \times \prod_{i=1}^{\ell} [p_1\psi_1(x_i)\xi_1(x_i) + p_2\psi_2(x_i)\xi_2(x_i)], \quad D = 0,1,\ldots,\ell-1, \\ \\ p_1^{b_1-1} p_2^{b_2-1} [p_1\xi_1(T) + p_2\xi_2(T)]^{(n-D)} \\ \times \prod_{i=1}^{D} [p_1\psi_1(x_i)\xi_1(x_i) + p_2\psi_2(x_i)\xi_2(x_i)], \quad D = \ell,\ldots,r-1, \\ \\ p_1^{b_1-1} p_2^{b_2-1} [p_1\xi_1(x_r) + p_2\xi_2(x_r)]^{(n-r)} \\ \times \prod_{i=1}^{r} [p_1\psi_1(x_i)\xi_1(x_i) + p_2\psi_2(x_i)\xi_2(x_i)], \quad D = r. \end{cases} \tag{3.3}$$

Similarly, the posterior densities for α_q and $\beta_q, q = 1,2$ are given, respectively, by

$$\pi^*(\alpha_q \mid p_1,\beta_1,\beta_2,x) \propto \begin{cases} \alpha_q^{\delta_q-1}\exp[-d_q\alpha_q][p_1\xi_1(x_\ell)+p_2\xi_2(x_\ell)]^{(n-\ell)} \\ \times\prod_{i=1}^{\ell}[p_1\psi_1(x_i)\xi_1(x_i)+p_2\psi_2(x_i)\xi_2(x_i)], & D=0,1,\ldots,\ell-1, \\ \alpha_q^{\delta_q-1}\exp[-d_q\alpha_q][p_1\xi_1(T)+p_2\xi_2(T)]^{(n-D)} \\ \times\prod_{i=1}^{D}[p_1\psi_1(x_i)\xi_1(x_i)+p_2\psi_2(x_i)\xi_2(x_i)], & D=\ell,\ldots,r-1, \\ \alpha_q^{\delta_q-1}\exp[-d_q\alpha_q][p_1\xi_1(x_r)+p_2\xi_2(x_r)]^{(n-r)} \\ \times\prod_{i=1}^{r}[p_1\psi_1(x_i)\xi_1(x_i)+p_2\psi_2(x_i)\xi_2(x_i)], & D=r. \end{cases} \tag{3.4}$$

$$\pi^*(\beta_q \mid p_1,\alpha_1,\alpha_2,x) \propto \begin{cases} \beta_q^{\delta_s-1}\exp[-d_s\beta_q][p_1\xi_1(x_\ell)+p_2\xi_2(x_\ell)]^{(n-\ell)} \\ \times\prod_{i=1}^{\ell}[p_1\psi_1(x_i)\xi_1(x_i)+p_2\psi_2(x_i)\xi_2(x_i)], & D=0,1,\ldots,\ell-1, \\ \beta_q^{\delta_s-1}\exp[-d_s\beta_q][p_1\xi_1(T)+p_2\xi_2(T)]^{(n-D)} \\ \times\prod_{i=1}^{D}[p_1\psi_1(x_i)\xi_1(x_i)+p_2\psi_2(x_i)\xi_2(x_i)], & D=\ell,\ldots,r-1, \\ \beta_q^{\delta_s-1}\exp[-d_s\beta_q][p_1\xi_1(x_r)+p_2\xi_2(x_r)]^{(n-r)} \\ \times\prod_{i=1}^{r}[p_1\psi_1(x_i)\xi_1(x_i)+p_2\psi_2(x_i)\xi_2(x_i)], & D=r. \end{cases} \tag{3.5}$$

where $s=3,4$.

The predictive PDF (2.8) can be written as

$$f^*(x_a^* \mid x) = \frac{\sum_{j=1}^{N} h_1(x_a^* \mid \theta_j, x)}{\sum_{j=1}^{N}\int_{x_r}^{\infty} h_1(x_a^* \mid \theta_j, x)\,dx_a^*}, \quad x_a^* > x_r, \tag{3.6}$$

where $\theta_j = p_j, \alpha_{1j}, \alpha_{2j}, \beta_{1j}, \beta_{2j}$, $j=1,2,3,\ldots,N$ are generated from the marginal posterior densities (3.3), (3.4) and (3.5),

$h_1(x_a^* \mid \theta, x) = [p_1\psi_1(x_a^*)\xi_1(x_a^*) + p_2\psi_2(x_a^*)\xi_2(x_a^*)]$

$\times\sum_{i=0}^{a-1}\omega_i^{(a)}[p_1\xi_1(x_a^*)+p_2\xi_2(x_a^*)]^{n-r-a+i}[p_1\xi_1(x_r)+p_2\xi_2(x_r)]^{-(n-r-a+i+1)}$, where, for $q=1,2,$

$\psi_q(z)=\alpha_q\beta_q z^{(\beta_q-1)}$, $\xi_q(z)=\exp[-\alpha_q z^{\beta_q}]$. A 100 τ% BPI (L,U) of the future observation X_a^* is given by solving the following two nonlinear equations

$$\frac{\sum_{j=1}^{N}\int_{L}^{\infty} h_1(x_a^* \mid \theta_j, x)\,dx_a^*}{\sum_{j=1}^{N}\int_{x_r}^{\infty} h_1(x_a^* \mid \theta_j, x)\,dx_a^*} = \frac{1+\tau}{2}, \tag{3.7}$$

$$\frac{\sum_{j=1}^{N}\int_{U}^{\infty} h_1(x_a^* \mid \theta_j, x)\,dx_a^*}{\sum_{j=1}^{N}\int_{x_r}^{\infty} h_1(x_a^* \mid \theta_j, x)\,dx_a^*} = \frac{1-\tau}{2}. \tag{3.8}$$

Upper record values

Also, for $urv's$ case, $m=-1$ ($k=1$), by multiplying the likelihood function (1.11) and the prior density function (3.1), the joint posterior density function will be in the form

$$\pi^*(\theta \mid x) \propto \begin{cases} p_1^{b_1-1} p_2^{b_2-1} \alpha_1^{\delta_1-1} \alpha_2^{\delta_2-1} \beta_1^{\delta_3-1} \beta_2^{\delta_4-1} \\ \times \exp\,[-(d_1\alpha_1 + d_2\alpha_2 + d_3\beta_1 + d_4\beta_2)] \\ \times [p_1\xi_1(x_\ell) + p_2\xi_2(x_\ell)] \\ \times \prod_{i=1}^{\ell} \frac{[p_1\psi_1(x_i)\xi_1(x_i) + p_2\psi_2(x_i)\xi_2(x_i)]}{[p_1\xi_1(x_i) + p_2\xi_2(x_i)]}, \quad D = 0,1,\ldots,\ell-1, \\ \\ p_1^{b_1-1} p_2^{b_2-1} \alpha_1^{\delta_1-1} \alpha_2^{\delta_2-1} \beta_1^{\delta_3-1} \beta_2^{\delta_4-1} \\ \times \exp\,[-(d_1\alpha_1 + d_2\alpha_2 + d_3\beta_1 + d_4\beta_2)] \\ \times \prod_{i=1}^{D} \frac{[p_1\psi_1(x_i)\xi_1(x_i) + p_2\psi_2(x_i)\xi_2(x_i)]}{[p_1\xi_1(x_i) + p_2\xi_2(x_i)]}, \quad D = \ell,\ldots,r-1, \\ \\ p_1^{b_1-1} p_2^{b_2-1} \alpha_1^{\delta_1-1} \alpha_2^{\delta_2-1} \beta_1^{\delta_3-1} \beta_2^{\delta_4-1} \\ \times \exp\,[-(d_1\alpha_1 + d_2\alpha_2 + d_3\beta_1 + d_4\beta_2)] \\ \times [p_1\xi_1(x_r) + p_2\xi_2(x_r)] \\ \times \prod_{i=1}^{r} \frac{[p_1\psi_1(x_i)\xi_1(x_i) + p_2\psi_2(x_i)\xi_2(x_i)]}{[p_1\xi_1(x_i) + p_2\xi_2(x_i)]}, \quad D = r. \end{cases} \tag{3.9}$$

From (3.9), the posterior density of p_1 is

$$\pi^*(p_1|\alpha_1,\alpha_2,\beta_1,\beta_2,x)\propto \begin{cases} p_1^{b_1-1}p_2^{b_2-1}[p_1\xi_1(x_\ell)+p_2\xi_2(x_\ell)] \\ \times\prod_{i=1}^{\ell}\frac{[p_1\psi_1(x_i)\xi_1(x_i)+p_2\psi_2(x_i)\xi_2(x_i)]}{[p_1\xi_1(x_i)+p_2\xi_2(x_i)]}, & D=0,1,\ldots,\ell-1, \\ p_1^{b_1-1}p_2^{b_2-1}\prod_{i=1}^{D}\frac{[p_1\psi_1(x_i)\xi_1(x_i)+p_2\psi_2(x_i)\xi_2(x_i)]}{[p_1\xi_1(x_i)+p_2\xi_2(x_i)]}, & D=\ell,\ldots,r-1, \\ p_1^{b_1-1}p_2^{b_2-1}[p_1\xi_1(x_r)+p_2\xi_2(x_r)] \\ \times\prod_{i=1}^{r}\frac{[p_1\psi_1(x_i)\xi_1(x_i)+p_2\psi_2(x_i)\xi_2(x_i)]}{[p_1\xi_1(x_i)+p_2\xi_2(x_i)]}, & D=r. \end{cases} \quad (3.10)$$

Similarly, the posterior densities for α_q and $\beta_q, q=1,2$ are given, respectively, by

$$\pi^*(\alpha_q\mid p_1,\beta_1,\beta_2,x)\propto \begin{cases} \alpha_q^{\delta_q-1}\exp\,[-d_q\alpha_q][p_1\xi_1(x_\ell)+p_2\xi_2(x_\ell)] \\ \times\prod_{i=1}^{\ell}\frac{[p_1\psi_1(x_i)\xi_1(x_i)+p_2\psi_2(x_i)\xi_2(x_i)]}{[p_1\xi_1(x_i)+p_2\xi_2(x_i)]}, & D=0,1,\ldots,\ell-1, \\ \alpha_q^{\delta_q-1}\exp\,[-d_q\alpha_q] \\ \times\prod_{i=1}^{D}\frac{[p_1\psi_1(x_i)\xi_1(x_i)+p_2\psi_2(x_i)\xi_2(x_i)]}{[p_1\xi_1(x_i)+p_2\xi_2(x_i)]}, & D=\ell,\ldots,r-1, \\ \alpha_q^{\delta_q-1}\exp\,[-d_q\alpha_q][p_1\xi_1(x_r)+p_2\xi_2(x_r)] \\ \times\prod_{i=1}^{r}\frac{[p_1\psi_1(x_i)\xi_1(x_i)+p_2\psi_2(x_i)\xi_2(x_i)]}{[p_1\xi_1(x_i)+p_2\xi_2(x_i)]}, & D=r. \end{cases} \quad (3.11)$$

$$\pi^*(\beta_q\mid p_1,\alpha_1,\alpha_2,x)\propto \begin{cases} \beta_q^{\delta_s-1}\exp\,[-d_s\beta_q][p_1\xi_1(x_\ell)+p_2\xi_2(x_\ell)] \\ \times\prod_{i=1}^{\ell}\frac{[p_1\psi_1(x_i)\xi_1(x_i)+p_2\psi_2(x_i)\xi_2(x_i)]}{[p_1\xi_1(x_i)+p_2\xi_2(x_i)]}, & D=0,1,\ldots,\ell-1, \\ \beta_q^{\delta_s-1}\exp\,[-d_s\beta_q] \\ \times\prod_{i=1}^{D}\frac{[p_1\psi_1(x_i)\xi_1(x_i)+p_2\psi_2(x_i)\xi_2(x_i)]}{[p_1\xi_1(x_i)+p_2\xi_2(x_i)]}, & D=\ell,\ldots,r-1, \\ \beta_q^{\delta_s-1}\exp\,[-d_s\beta_q][p_1\xi_1(x_r)+p_2\xi_2(x_r)] \\ \times\prod_{i=1}^{r}\frac{[p_1\psi_1(x_i)\xi_1(x_i)+p_2\psi_2(x_i)\xi_2(x_i)]}{[p_1\xi_1(x_i)+p_2\xi_2(x_i)]}, & D=r. \end{cases} \quad (3.12)$$

where $s=3,4.$

The predictive PDF (2.8) can be written as

$$f^*(x_a^* \mid x) = \frac{\sum_{j=1}^{N} h_2(x_a^* \mid \theta_j, x)}{\sum_{j=1}^{N} \int_{x_r}^{\infty} h_2(x_a^* \mid \theta_j, x) dx_a^*}, \quad x_a^* > x_r, \quad (3.13)$$

where $\theta_j,\ j = 1,2,3,\ldots,N$ are generated from the marginal posterior densities (3.10), (3.11), (3.12) and

$$h_2(x_a^* \mid \theta, x) = \frac{[p_1\psi_1(x_a^*)\xi_1(x_a^*) + p_2\psi_2(x_a^*)\xi_2(x_a^*)]}{[p_1\xi_1(x_r) + p_2\xi_2(x_r)]}$$

$$\times \sum_{i=0}^{a-1} \omega_i^{(a)} (\ln[p_1\xi_1(x_a^*) + p_2\xi_2(x_a^*)])^i (\ln[p_1\xi_1(x_r) + p_2\xi_2(x_r)])^{a-i-1}.$$

A $100\,\tau\%$ BPI (L,U) of the future observation X_a^* is given by solving the following two nonlinear equations

$$\frac{\sum_{j=1}^{N} \int_{L}^{\infty} h_2(x_a^* \mid \theta_j, x) dx_a^*}{\sum_{j=1}^{N} \int_{x_r}^{\infty} h_2(x_a^* \mid \theta_j, x) dx_a^*} = \frac{1+\tau}{2}, \quad (3.14)$$

$$\frac{\sum_{j=1}^{N} \int_{U}^{\infty} h_2(x_a^* \mid \theta_j, x) dx_a^*}{\sum_{j=1}^{N} \int_{x_r}^{\infty} h_2(x_a^* \mid \theta_j, x) dx_a^*} = \frac{1-\tau}{2}. \quad (3.15)$$

Numerical Computations

In this section, 95% and 99% BPI's for future observations from a mixture of two $Weibull(\alpha_j, \beta_j),\ j = 1,2,$ components are obtained by considering one sample scheme.

Simulated results

Here, we generate data from a mixture of two $Weibull(\alpha_j, \beta_j)$ based on gos's under generalized Type I HCS.

Upper order statistics

The 95% and 99% $BPI's$ for $X_a^*, a=1,2,3$ are obtained according to the following steps:

1. For given values of the prior parameters (b_1, b_2), generate a random value p from the $Beta(b_1, b_2)$ distribution.
2. For given values of the prior parameters δ_i, d_i for $i=1,2$, generate a random value α_i from the $Gamma(\delta_i, d_i)$ distribution.
3. For a given values of the prior parameters δ_s, d_s for $s=3,4$, generate a random value β_i for $i=1,2$, from the $Gamma(\delta_s, d_s)$ distribution.
4. Using the generated values of p, α_1, α_2, β_1 and β_2, we generate ordered sample of size n from a mixture of two $Weibull(\alpha_i, \beta_i)$, $i=1,2$, components as follows:

- Generate two observations u_1, u_2 from Uniform $(0,1)$.
- if $u_1 \le p$, then $x=[-\frac{\ln(1-u_2)}{\alpha_1}]^{\frac{1}{\beta_1}}$, otherwise $x=[-\frac{\ln(1-u_2)}{\alpha_2}]^{\frac{1}{\beta_2}}$.
- Repeat the above steps n times to get a sample of size n.

5. The above generated sample was censored using generalized Type-I HCS (and special case from it).
6. Generate ($p_j, \alpha_{1j}, \alpha_{2j}, \beta_{1j}, \beta_{2j}$), $j=1,2,\ldots,1000$ from the posterior densities (3.3), (3.4) and (3.5) using MCMC algorithm.
7. The 95% and 99% $BPI's$ for the future uos's are obtained by solving numerically, equations (3.7) and (3.8) with $\tau=0.95$ and $\tau=0.99$.

Upper record values

In this case the steps are:

1. We generate the parameters as in the case of $uos's$.
2. Using the generated values of p, α_1, α_2, β_1 and β_2, we generate upper record values of size n from a mixture of two $Weibull(\alpha_i, \beta_i)$, $i=1,2$, components.
3. The above generated sample was censored using generalized Type-I HCS (and special case from it).
4. Generate ($p_j, \alpha_{1j}, \alpha_{2j}, \beta_{1j}, \beta_{2j}$), $j=1,2,\ldots,1000$ from the posterior densities (3.10), (3.11) and (3.12) using MCMC algorithm.
5. The 95% and 99% $BPI's$ for the future urv's are obtained by solving numerically, equations (3.14) and (3.15) with $\tau=0.95$ and $\tau=0.99$.

The 95% and 99% $BPI's$ for future observations X_a^*, $a=1,2,3$ based on $uos's$ and $urv's$ under generalized Type-I HCS (and special case from it) are displayed in Tables

(1a,b), (2a,b), (3a,b) and (4a,b). Numerical results are listed in Tables (1a,b) and (2a,b) taking into the hyper parameters $b_1 = 2, b_2 = 3, d_1 = 3.5, d_2 = 2.8, d_3 = 1.6, d_4 = 0.3,$ $\delta_1 = 3.6, \delta_2 = 2.5, \delta_3 = 2, \delta_4 = 0.4$. While, the numerical results are listed in Tables (3a,b) and (4a,b) taking into the hyper parameters $b_1 = 2, b_2 = 3, d_1 = 1.5, d_2 = 1.8, d_3 = 1.6, d_4 = 3.3,$ $\delta_1 = 3.6, \delta_2 = 3.5, \delta_3 = 2, \delta_4 = 3.4$. The number of samples which cover the $BPI's$ is 10000 samples.

Table(1a): Case of generalized Type-I HCS ($uos's$)

(n,r)	X_a^*	95%			99%		
(ℓ,T)		(L,U)	Length	CP(%)	(L,U)	Length	CP(%)
(15, 9) (7, 0.2)	X_1^*	(0.26268, 1.92313)	1.66045	92.92	(0.25777, 3.15983)	2.90206	93.51
	X_2^*	(0.32408, 3.94701)	3.62293	95.84	(0.28429, 6.28461)	6.00032	96.09
	X_3^*	(0.34206, 5.4692)	5.12714	93.1	(0.29171, 8.99105)	8.69934	95.82
(35, 32) (30, 0.5)	X_1^*	(0.6134, 5.42678)	4.81338	97.67	(0.60864, 17.8975)	17.2888	97.94
	X_2^*	(0.67944, 40.2431)	39.5636	97.94	(0.63811, 152.107)	151.468	99.03
	X_3^*	(0.73629,325.576)	324.839	96.76	(0.56633, 522.17)	521.603	98.15
(80, 77) (74, 0.7)	X_1^*	(0.91179, 2.12161)	1.20982	93.44	(0.90756, 4.74582)	3.83826	95.45
	X_2^*	(0.85345, 14.0889)	13.2354	97.85	(0.88221, 40.7491)	39.8668	96.15
	X_3^*	(0.81583, 118.062)	117.246	96.93	(0.86311, 332.843)	331.979	95.67

Table (1b): $\ell = 0$ (Case of Type-I HCS) ($uos's$)

(n,r)	X_a^*	95%			99%		
(T)		(L,U)	Length	CP(%)	(L,U)	Length	CP(%)
(15, 9) (0.2)	X_1^*	(0.26337, 2.3356)	2.07223	92.88	(0.2579, 4.01287)	3.75497	93.51
	X_2^*	(0.33305, 5.11936)	4.78631	95.56	(0.28759, 8.56204)	8.27445	95.83
	X_3^*	(0.35189, 7.33144)	6.97955	95.42	(0.29527, 12.7967)	12.5014	98.73
(35, 32) (0.5)	X_1^*	(0.61971, 4.05604)	3.43633	97.33	(0.61003, 6.87509)	6.26506	97.87
	X_2^*	(0.75624, 11.1483)	10.3920	99.7	(0.66875, 20.66)	19.9912	97.96
	X_3^*	(0.87699, 37.6668)	36.7898	96.14	(0.71583, 84.1115)	83.3956	97.12
(80, 77) (0.7)	X_1^*	(0.91222, 3.57025)	2.65803	94.27	(0.90764, 10.4293)	9.52166	95.45
	X_2^*	(0.84991, 18.2921)	17.4421	98.06	(0.9361, 97.619)	96.6829	98.24
	X_3^*	(0.81072, 292.031)	291.220	97.94	(0.86027, 142.22)	141.359	97.63

Table (2a): Case of generalized Type-I HCS ($urv's$)

(n,r) (T)	X_a^*	95%			99%		
		(L,U)	Length	CP(%)	(L,U)	Length	CP(%)
(8, 5) (0.6)	X_1^*	(0.91060, 354.765)	353.854	92.0	(0.92743, 136.3)	135.372	93.0
	X_2^*	(1.85263, 551.04)	549.187	92.32	(1.19296, 371.228)	370.035	92.0
	X_3^*	(4.59296, 732.42)	727.827	91.74	(1.93041, 739.79)	737.859	91.0
(10, 7) (0.8)	X_1^*	(1.02801, 54.8772)	53.8491	95.81	(0.97550, 121.767)	120.791	97.05
	X_2^*	(1.684, 168.75)	167.066	91.66	(1.24596, 305.7)	304.454	95.5
	X_3^*	(3.20071, 374.729)	371.528	92.75	(2.01714, 584.75)	582.732	93.47
(13, 10) (0.9)	X_1^*	(2.10301, 36.7516)	34.6485	98.7	(2.03856, 82.7784)	80.7398	99.2
	X_2^*	(2.85983, 84.8606)	82.0007	99.0	(2.36176, 200.844)	198.482	99.5
	X_3^*	(4.42779, 153.451)	149.023	98.1	(3.22527, 399.725)	396.499	98.3

(n,r) (ℓ,T)	X_a^*	95%			99%		
		(L,U)	Length	CP(%)	(L,U)	Length	CP(%)
(8, 5) (3, 0.6)	X_1^*	(0.89625, 94.1962)	93.2999	91.19	(0.84089, 46.393)	45.5521	93.0
	X_2^*	(1.6379, 285.294)	283.656	92.55	(1.1376, 331.01)	329.872	93.54
	X_3^*	(3.5368, 580.716)	577.179	91.8	(2.03376, 659.68)	657.646	92.13
(10, 7) (5, 0.8)	X_1^*	(1.02189, 29.4238)	28.4019	96.03	(0.97375, 70.6837)	69.7099	97.09
	X_2^*	(1.59862, 65.99)	64.3913	92.6	(1.21477, 176.308)	175.093	96.03
	X_3^*	(2.87094, 112.419)	109.548	90.26	(1.88066, 357.513)	355.632	94.22
(13, 10) (7, 0.9)	X_1^*	(2.08825, 20.4263)	18.3380	91.0	(2.03566, 43.5135)	41.4778	97.2
	X_2^*	(2.68716, 38.0065)	35.3193	92.7	(2.29727, 101.224)	98.9267	98.0
	X_3^*	(3.85806, 58.876)	55.0179	90.9	(2.96017, 206.947)	203.986	98.55

Table(2b): $\ell = 0$ (Case of Type-I HCS) ($urv's$)

Table (3a): Case of generalized Type-I HCS ($uos's$)

(n,r) (ℓ,T)	X_a^*	95% (L,U)	Length	CP(%)	99% (L,U)	Length	CP(%)
(15, 9) (7, 0.2)	X_1^*	(0.26045, 1.28905)	1.0286	97.0	(0.25733, 2.11256)	1.85523	97.0
	X_2^*	(0.29751, 2.48588)	2.18837	97.0	(0.27366, 4.04325)	3.76959	99.0
	X_3^*	(0.37698, 4.40007)	4.02309	92.0	(0.31936, 7.1829)	6.86354	97.0
(35, 32) (30, 0.5)	X_1^*	(0.61113, 5.81212)	5.20099	97.0	(0.60812, 13.8823)	13.2741	97.0
	X_2^*	(0.64577, 24.15)	23.5042	97.0	(0.62246, 58.0404)	57.4179	98.0
	X_3^*	(0.7418, 135.531)	134.789	99.0	(0.67275, 348.68)	348.007	100.0
(80, 77) (74, 0.7)	X_1^*	(0.91165, 2.00409)	1.09244	95.0	(0.90747, 2.9062)	1.99873	96.0
	X_2^*	(0.85531, 9.41401)	8.5587	98.0	(0.88365, 18.4523)	17.5686	97.0
	X_3^*	(1.13092, 98.4272)	97.2962	97.0	(1.01451, 215.827)	214.812	97.0

(n,r) (T)	X_a^*	95% (L,U)	Length	CP(%)	99% (L,U)	Length	CP(%)
(15, 9) (0.2)	X_1^*	(0.26208, 1.57476)	1.31268	97.0	(0.25759, 2.77145)	2.51386	97.0
	X_2^*	(0.31684, 2.98931)	2.67247	98.0	(0.28110, 5.48183)	5.20073	99.0
	X_3^*	(0.44185, 5.13515)	4.6933	95.0	(0.34621, 10.2822)	9.93599	98.0
(35, 32) (0.5)	X_1^*	(0.61262, 5.61746)	5.00484	97.0	(0.60838, 13.4804)	12.8720	97.0
	X_2^*	(0.66302, 23.567)	22.9039	95.0	(0.63035, 55.7968)	55.1664	98.0
	X_3^*	(0.80826, 129.16)	128.351	95.0	(0.70324, 333.843)	333.139	100.0
(80, 77) (0.7)	X_1^*	(0.90993, 1.65675)	0.74682	91.0	(0.90736, 3.405)	2.49764	92.0
	X_2^*	(0.94568, 3.35621)	2.41053	92.0	(0.92782, 10.4607)	9.53288	96.0
	X_3^*	(1.03906, 11.7153)	10.6762	94.0	(1.00065, 45.2245)	44.2238	98.0

Table(3b): $\ell = 0$ (Case of Type-I HCS) ($uos's$)

Table(4a): Case of generalized Type-I HCS ($urv's$)

(n,r) (T)	X_a^*	95%			99%		
		(L,U)	Length	CP(%)	(L,U)	Length	CP(%)
(8, 5) (0.6)	X_1^*	(0.86372, 21.8715)	21.0077	93.0	(0.83301, 59.0629)	58.2298	93.0
	X_2^*	(1.21441, 56.8909)	55.6764	84.0	(0.98062, 203.183)	202.202	93.0
	X_3^*	(1.94521, 127.185)	125.239	83.0	(1.36832, 482.222)	480.853	89.0
(10, 7) (0.8)	X_1^*	(0.98234, 8.12263)	7.14029	96.0	(0.96658, 17.9838)	17.0172	98.0
	X_2^*	(1.14947, 16.7277)	15.5782	97.0	(1.04004, 39.4713)	38.4312	98.0
	X_3^*	(1.44716, 29.5217)	28.0745	98.0	(1.21139, 66.9926)	65.7812	98.0
(13, 10) (0.9)	X_1^*	(2.038, 7.21073)	5.17273	92.0	(2.02428, 7.21945)	5.19517	92.0
	X_2^*	(1.91319, 11.7885)	9.87531	95.0	(2.06229, 11.4372)	9.37491	95.0
	X_3^*	(2.35648, 16.8839)	14.5274	95.0	(2.15196, 16.2069)	14.0549	94.0

Table(4b): $\ell = 0$ (Case of Type-I HCS) ($urv's$)

(n,r) (ℓ,T)	X_a^*	95%			99%		
		(L,U)	Length	CP(%)	(L,U)	Length	CP(%)
(8, 5) (3, 0.6)	X_1^*	(0.86702, 17.1298)	16.2627	93.0	(0.83342, 32.8619)	32.0284	93.0
	X_2^*	(1.25579, 34.9464)	33.6906	83.0	(0.99804, 65.0798)	64.0817	91.0
	X_3^*	(2.06961, 58.1282)	56.0585	82.0	(1.43926, 109.494)	108.054	86.0
(10, 7) (5, 0.8)	X_1^*	(1.00254, 12.5435)	11.5409	98.0	(0.97033, 22.6025)	21.6321	98.0
	X_2^*	(1.35622, 23.4759)	22.1196	95.0	(1.12542, 41.5166)	40.3911	97.0
	X_3^*	(2.04391, 36.8675)	34.8235	94.0	(1.51733, 63.8655)	62.3481	97.0
(13, 10) (7, 0.9)	X_1^*	(2.07121, 12.3062)	10.2349	92.0	(2.0307, 18.8328)	16.8021	97.0
	X_2^*	(2.50572, 19.7326)	17.2268	99.0	(2.20149, 31.5358)	29.3343	99.0
	X_3^*	(3.31201, 27.472)	24.1599	94.0	(2.63081, 46.7427)	44.1118	99.0

Real data results

In this subsection, to illustrate the prediction results, let us consider the data given by Razali and Salih [33] consisting of ordered lifetimes of 20 electronic components, which from a mixture of two $Weibull(\alpha,\beta)$ distributions. Its elements are shown as follows 0.03, 0.12, 0.22, 0.35, 0.73, 0.79, 1.25, 1.41, 1.52, 1.79, 1.8, 1.94, 2.38, 2.4, 2.87, 2.99, 3.14, 3.17, 4.72 and 5.09.

We shall use these data to consider three different generalized Type-I HCS's:
1. When $r = 17$, $\ell = 15$ and $T = 0.7$. Since $x_{15:20} > T$, the testing would have terminated in this case at time $x_{15:20} = 2.87$. 2. When $r = 17$, $\ell = 15$ and $T = 3$. Since $x_{15:20} < T < x_{17:20}$, the testing would have terminated in this case at time T. 3. When $r = 17$, $\ell = 15$ and $T = 4$. Since $x_{15:20} < x_{17:20} < T$, the testing would have terminated in this case at time $x_{17:20} = 3.14$. we then used the equations presented earlier in Section 3.1 to construct 95% and 99% one-sample *BPI's* for future order statistics X_a^*, $a = 1,2,3$, from the same sample. The results displayed in Tables (5a, b, c) and (6a, b, c).

(n,r) (ℓ,T)	X_a^*	95%		99%	
		(L,U)	Length	(L,U)	Length
(20, 17) (15, 4)	X_1^*	(3.16313, 6.96493)	3.8018	(3.14458, 8.89835)	5.75377
	X_2^*	(3.41801, 11.1376)	7.71959	(3.26026, 14.6795)	11.4192
	X_3^*	(4.13034, 21.8827)	17.7523	(3.67177, 31.1687)	27.4969

Table (5a): The case 1 (the experiment is terminated at time $x_{15:20} = 2.87$) the hyper parameters
$b_1 = 2, b_2 = 3, d_1 = 3.5, d_2 = 2.8, d_3 = 1.6,\ d_4 = 0.3, \delta_1 = 3.6, \delta_2 = 2.5, \delta_3 = 2, \delta_4 = 0.4$
Table(5b): The case 2 (the experiment is terminated at time T) the hyper parameters
$b_1 = 2, b_2 = 3, d_1 = 3.5, d_2 = 2.8, d_3 = 1.6,\ d_4 = 0.3, \delta_1 = 3.6, \delta_2 = 2.5, \delta_3 = 2, \delta_4 = 0.4$

(n,r) (ℓ,T)	X_a^*	95%		99%	
		(L,U)	Length	(L,U)	Length
(20, 17) (15, 0.7)	X_1^*	(3.1623, 7.3071)	4.1448	(3.14436, 9.89486)	6.7505
	X_2^*	(3.39681, 13.0554)	9.65859	(3.2504, 19.512)	16.2616
	X_3^*	(4.06013, 34.5788)	30.5186	(3.62429, 61.9106)	58.2863

Table(5c): The case 3 (the experiment is terminated at time $x_{17:20} = 3.14$) the hyper parameters

(n,r) (ℓ,T)	X_a^*	95%		99%	
		(L,U)	Length	(L,U)	Length
(20, 17) (15, 0.7)	X_1^*	(3.16709, 7.78604)	4.61895	(3.14536, 10.3606)	7.21524
	X_2^*	(3.46199, 13.2909)	9.82891	(3.27415, 19.2433)	15.9691
	X_3^*	(4.29952, 28.9123)	24.6127	(3.75624, 57.6384)	53.8821

Table(6a): The case 1 (the experiment is terminated at time $x_{15:20} = 2.87$) the hyper parameters

$$b_1 = 2, b_2 = 3, d_1 = 1.5, d_2 = 1.8, d_3 = 1.6,\ d_4 = 3.3, \delta_1 = 3.6, \delta_2 = 3.5, \delta_3 = 2, \delta_4 = 3.4$$

(n,r) (ℓ,T)	X_a^*	95% (L,U)	95% Length	99% (L,U)	99% Length
(20, 17) (15, 3)	X_1^*	(3.16182, 7.28919)	4.12737	(3.14411, 10.2076)	7.06349
	X_2^*	(3.39712, 13.532)	10.1348	(3.24727, 21.6277)	18.3804
	X_3^*	(4.06244, 35.4001)	31.3376	(3.62515, 64.5104)	60.8852

Table(6b): The case 2 (the experiment is terminated at time T) the hyper parameters
$$b_1 = 2, b_2 = 3, d_1 = 1.5, d_2 = 1.8, d_3 = 1.6,\ d_4 = 3.3, \delta_1 = 3.6, \delta_2 = 3.5, \delta_3 = 2, \delta_4 = 3.4$$

Table (6c): The case 3 (terminated the experiment at time $x_{17:20} = 3.14$) the hyper parameters

(n,r) (ℓ,T)	X_a^*	95% (L,U)	95% Length	99% (L,U)	99% Length
(20, 17) (15, 3)	X_1^*	(3.16491, 7.25952)	4.09461	(3.14492, 9.36202)	6.2171
	X_2^*	(3.43354, 11.8356)	8.40206	(3.2645, 15.7116)	12.4471
	X_3^*	(4.19733, 23.8396)	19.6422	(3.70252, 33.9814)	30.2788

(n,r) (ℓ,T)	X_a^*	95% (L,U)	95% Length	99% (L,U)	99% Length
(20, 17) (15, 4)	X_1^*	(3.15972, 6.98084)	3.82112	(3.14402, 9.76497)	6.62095
	X_2^*	(3.36752, 12.8271)	9.45958	(3.23718, 20.7814)	17.5442
	X_3^*	(3.94534, 33.2597)	29.3143	(3.56246, 62.5452)	58.9827

$$b_1 = 2, b_2 = 3, d_1 = 1.5, d_2 = 1.8, d_3 = 1.6,\ d_4 = 3.3, \delta_1 = 3.6, \delta_2 = 3.5, \delta_3 = 2, \delta_4 = 3.4$$

Conclusions

1. Bayesian prediction intervals for future observations are obtained using a one-sample scheme based on a finite mixture of two Weibull components model from gos's under generalized Type I HCS. Our results are specialized to uos's and urv's. Also, we used real data example. 2. It is evident from Tables (1) and (3) that, the lengths of the BPI increase as the sample size increases. While, from Tables (2) and (4), the lengths of the BPI decrease as the sample size increases. 3. It is evident from all tables that the lower bounds are relatively insensitive to the specification of the hyper parameters while, the upper bounds are somewhat sensitive. 4. In general, for fixed sample size n and fixed censored sizes r, ℓ and T, the length of the BPI increase by increasing a.

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