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Coupled Fixed Point Theorems for Weak Contractions on Partial Metric Space

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Abstract: *In this paper, we give anew weak contraction mapping and by using this contraction mapping we establish Coupled fixed point theorems in partial metric space. Our result extends some known results duo to Hassen Aydi, Erdal Karapinar and Wasfi Shatanawi [3].*

Keywords: *coupled fixed point, partial metric space, mixed monotone mapping.*

Introduction

Recently, studies on the existence and uniqueness of fixed points of self-mappings on partial metric spaces have gained momentum (see [2], [3], [9]). The idea of partial metric space, a generalization of metric space, was introduced by Mathews [12] in 1992. When compared to metric spaces, the innovation of partial metric spaces is that the self distance of a point is not necessarily zero [13]. This feature of partial metrics makes them suitable for many purposes of semantics and domain theory in computer sciences. In particular, partial metric spaces have applications on the Scott-Strachey order-theoretic topological models [18] used in the logics of computer programs.The notation of coupled fixed point was introduced by Chang and Ma [6]. since then, the concept has been of interest to many researchers in metrical fixed point theory. Baskar and Lakshmikantham [5] introduced the concepts of coupled fixed point and mixed monotone property for contractive operators of the form F: X ×X   X, where X is partially ordered metric space, and then established some interesting coupled fixed point theorems. They also illustrated these important results by proving the existence and uniqueness of the solution for a periodic boundary value problem. The result of [5] has also been generalized and extended by Likshmantham and Ciric [10].For more details on coupled fixed point theory, we also refer the reader to [5],[7],[4],[14],[15],[16],[17]. In [16] Sabetghadam, Masiha and Sanatpour extended the result of Bhashkar and Lakshmikantham [5] by considering the contraction condition. $d(F(x, y), F(u, v)) = kd(x, u)+ ld(y, v)$ where k, l are nonnegative constants with $k + l < 1$. definitions and properties of coupled fixed point and partial metric space

Definition 1.1. A partial metric on a nonempty set X is a function $p : X \times X \longrightarrow R+$
such that for all x, y, z ∈ X
(p1) x = y   p(x, x)= p(x, y) = p(y, y),
(p2) p(x, x) = p(x, y),
(p3) p(x, y) = p(y, x),

(p4) p(x, y) = p(x, z) + p(z, y)- p(z, z),
A partial metric space is a pair (X, p) such that X is a nonempty set and p is a Partial metric on X.

Remark 1.1. It is clear that , if p(x, y) = 0, then from (p1), (p2) and (p3), x =y. But if x ≠y, p(x, y) may not 0. Each partial metric p on X generates a T0 topology ¿p on X which has as a base the family of open p-ball.

If p is a partial metric on X,then the function

$p^s : X \times X \to \mathbb{R}_+$ *given by*

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y),$$

is a metric space on X.
Example 1.1. (see e,g.[12],[2][9]). Consider
X =R+ with p(x, y) = max{x, y}.Then (R+, p) is a partial metric

space. It is clear that p is not a (usual) metric. Note that in this case $p^s(x,y) = |x-y|$.

Example 1.2. (see[8]). Let

$X = \{[a,b] : a,b \in \mathbb{R}, a \le b\}$ and define p

p([a,b],[c,d]) =max{b,d}- min{a, c}, Then,(X, p) is a partial metric space.

Definition 1.2. Let(X, p) be a partial metric space and {xn} be a sequence in X.Then

(i) {xn} converges to a point x 2 X if and only if

$p(x,x) = \lim_{n,m\to+\infty} p(x,x_n)$.

(ii) {xn} is called a Cauchy sequence if their exists (and is finite)

$\lim_{n,m\to+\infty} p(x_n,x_m)$.

Definition 1.3. A partial metric space (X, p) is said to be complete if every Cauchy sequence {xn} in X converges , with respect to *T*p.to a point

$x \in X$ such that

$p(x,x) = \lim_{n,m\to+\infty} p(x_n,x_m)$.

Lemma 1.1. Let (X, p) be a partial metric space .Then (a){xn} is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, ps), (b)(X, p) is complete if an only if the metric space (X, ps) is complete.Furthermore ,$\lim_{n\to+\infty} p^s(x_n,x)=0$, if and only if $p(x,x)=\lim_{n\to+\infty} p(x_n,x)=\lim_{n,m\to+\infty} p(x_n,x_m)$

Definition 1.4. (Bhashkar and Lakshmikantham[5]).An element $(x,y)\in X\times X$ is called a coupled fixed point of mapping $F:X\times X\to X$ $if\, x=F(x,y)$ $and\, y=F(y,x).$

Definition 1.5. Let (X, p) be a partial metric. We endow $(x,y)\in X\times X$ with the partial metric v defined for $(x,y),(u,v)\in X\times X$ by $v((x,y),(u,v))=p(x,u)+p(y,v).$

A mapping $F:X\times X\to X$ is said to be continuous at $(x,y)\in X\times X$, if for every $\varepsilon>0,$ there exist $\delta>0$ such that $F(B_v((x,y),\delta))\subseteq B_v(F(x,y),\varepsilon).$

Before presenting our main results,we recall some basic concepts.

Definition1.6.(Bhashkar and Lakshmikantham[5]).Let (X,≤) be a partial ordered set and $F:X\times X\to X.$ mapping F is said to has the mixed monotone property if $x_1,x_2\in X, x_1\le x_2\Rightarrow F(x_1,y)\le F(x_2,y)$ $for\ any\ y\in X,$ And $y_1,y_2\in X, y_1\le y_2\Rightarrow F(x,y_1)\le F(x,y_2)$ $for\ any\ x\in X,$
Firstly Bhashkar and Lakshmikantham [6] proved the following result.

Theorem 1.1. (Bhashkar and Lakshmikantham [5]).Let (X,,≤) be partial ordered set and suppose there is a partial metric P on X such that (X, p) is a complete partial metric space. Let $F:X\times X\to X.$ be a mapping having the mixed monotone on X. Assume that there exist a $k\in[0,1)$ with $d(F(x,y),F(u,v))\le\frac{k}{2}[d(x,u)+d(y,v)]$ for all $x,y,u,v\in X$ with x ≥ u and y ≤v. suppose either F is continuous or X has the following properties:-
(1)if a non-decreasing $x_n\to x$,then xn ≤ x for all n.
(2)if a non-increasing $x_n\to x$,then xn ≥ x for all n.
if there exist x0, y0 $\in$ X such that $x_0\le f(x_0,y_0)$ and $y\ge f(y_0,x_0)$ then there $x,y\in X$ such that $x=f(x,y)$ and $y=f(y,x)$,that is, F has a coupled fixed point.
After that, Luong and Thuan[11] obtained a more general result ,For this ,let Φ denoted all function $\varphi:[0,+\infty)\to[0,+\infty)$ which satisfy
(i) φ *is continuous and non-decreasing,*
(ii) $\varphi(t)=0$ *if and only if* $t=0,$
(iii) $\varphi(t+s)\le\varphi(t)+\varphi(s),\ \forall t,s\in[0,+\infty).$
Again, Let Ψ denoted all function $\Psi:[0,+\infty)\to[0,+\infty)$ which satisfy $\lim_{t\to r}\psi(t)>0$ for all $r>0$ and $\lim_{t\to 0^+}\psi(t)=$ 0. It's an easy matter to see the following note.
Remark 1.2. $\Phi\in\Psi.$
Remark 1.3. For any $t\in[0,+\infty)$ we have $\frac{1}{2}\varphi(t)\le\varphi(\frac{t}{2}).$
Now ,We state the main result of Luong and Thuan[11]:

Theorem 1.2. Let (X,≤) be partial ordered set and suppose there is a partial metric P on X such that (X, p) is a complete partial metric space. Let $F: X \times X \to X$ be a mapping having the mixed monotone on X.Assume that there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that

$$\varphi(d(F(x,y),F(u,v))) \le \frac{1}{2}\varphi(d(x,u)+d(y,v)) - \psi\left(\frac{d(x,u)+d(y,v)}{2}\right)$$

for all $x, y, u, v \in X$ with x ≥ u and y ≤v. suppose either F is continuous or X has the following properties:-
(1)if a non-decreasing $x_n \to x$,then xn ≤ x for all n.
(2)if a non-increasing $x_n \to x$,then xn ≥ x for all n.
if there exist x0, y0 ∈ X such that $x_0 \le f(x_0, y_0)$ and $y \ge f(y_0, x_0)$ then there $x, y \in X$ such that $x = f(x, y)$ and $y = f(y, x)$,that is, F has a coupled fixed point.
Remark 1.4. Let $k \in [0,1)$.Taking $\varphi(t) = t$ and $\psi(t) = (1-k)(t)$ in Theorem 2.1., we obtain Theorem 1.1.
Resentaly, Hassen Aydi, Erdal Karapinar and Wasfi Shatanawi[3]. introduced more general contraction
condition which generalized the results duo to Luong and Thuan[12], and there result in the following theorems

Theorem 1.3. Let (X,≤) be partial ordered set and suppose there is a partial metric P on X such that (X, p) is a complete partial metric space. Let $F: X \times X \to X$ be a mapping having the mixed monotone on X.Assume that there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that

$$\varphi(d(F(x,y),F(u,v))) \le \varphi\left(\frac{d(x,u)+d(y,v)}{2}\right) - \psi\left(\frac{d(x,u)+d(y,v)}{2}\right)$$

for all $x, y, u, v \in X$ with x ≥ u and y ≤v. suppose either F is continuous or X has the following properties:
(1)if a non-decreasing $x_n \to x$,then xn ≤ x for all n.
(2)if a non-increasing $x_n \to x$,then xn ≥ x for all n.
if there exist x0, y0 ∈ X such that $x_0 \le f(x_0, y_0)$ and $y \ge f(y_0, x_0)$ then there $x, y \in X$ such that $x = f(x, y)$ and $y = f(y, x)$,that is, F has a coupled fixed point.

Now, we give anew weak contraction condition which generalized the previous results. Also, we prove
some coupled fixed point theorems on ordered partial metric space by using this weak contraction condition.
Finally we introduce an application to support our results.

The main results

The aim of this work is to prove the following theorem.

Theorem 2.1. Let (X,≤) be partial ordered set and suppose there is a partial metric P on X such that (X, p) is a complete partial metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone on X.Assume that there exist $\varphi \in \Phi$ *and* $\psi \in \Psi$ such that

$$\varphi(p(F(x,y),F(u,v))) \le \varphi\left(\frac{M(x,u)+M(y,v)}{2}\right) - \psi\left(\frac{M(x,u)+M(y,v)}{2}\right) \quad (1)$$

Since

$$M(x,u) = max\{p(x,u),p(x,F(x,y)),p(u,F(u,v)),\frac{1}{2}[p(u,F(x,y))+p(x,F(u,v))]\}$$

And $M(y,v) = max\{p(y,v),p(y,F(y,x)),p(v,F(v,u)),\frac{1}{2}[p(v,F(y,x))+p(y,F(v,u))]\}$

for all $x,y,u,v \in X$ with x ≥ u and y ≤v. suppose either F is continuous or X has the following properties:
(1)if a non-decreasing $x_n \to x$,then xn ≤ x for all n.
(2)if a non-increasing $x_n \to x$,then xn ≥ x for all n.
if there exist x0, y0 ∈ X such that $x_0 \le f(x_0,y_0)$ and $y \ge f(y_0,x_0)$ then there $x,y \in X$ such that $x = f(x,y)$ and $y = f(y,x)$,that is, F has a coupled fixed point. Furtharmor $p(x,x) = p(y,y) = 0$

proof. Since $x_0 \le f(x_0,y_0) = x_1$ (say) and $y_0 \ge f(y_0,x_0) = y_1$ (say), Letting $x_2 = f(x_1,y_1)$ and $y_2 = f(y_1,x_1)$.

$$f^2(x_0,y_0) = f(f(x_0,y_0),f(y_0,x_0)) = f(x_1,y_1) = x^2.$$
$$f^2(y_0,x_0) = f(f(y_0,x_0),f(x_0,y_0)) = f(y_1,x_1) = y^2.$$

We Now have, due to the mixed monotone property of F.
$x_2 = f(x_1,y_1) \ge f(x_0,y_0) = x_1$ and
$y_2 = f(y_1,x_1) \ge f(y_0,x_0) = y_1.$ further,for $n = 1,2,3,\ldots$, we let

$$x_{n+1} = f^{n+1}(x_0,y_0) = f(f^n(x_0,y_0),f^n(y_0,x_0)).$$

And $y_{n+1} = f^{n+1}(y_0,x_0) = f(f^n(y_0,x_0,f^n(x_0,y_0)).$

We can easily verify that
$$x_0 \le f(x_0,y_0) = x_1 \le f(x_1,y) = x_2 \le \ldots \le f^{n+1}(x_0,y_0) = x_{n+1}.$$
$$y_0 \ge f(y_1,x_1) = y_1 \ge f(y_1,x_1) = y_2 \ge \ldots \ge f^{n+1}(y_0,x_0) = y_{n+1}.$$

Since $x_n \ge x_{n+1}$ and $y_n \le y_{n+1}$ frome (1), we have

$$\begin{aligned}\phi(p(x_n,x_{n+1})) &= \varphi(p(F(x_{n-1},y_{n-1}),F(x_n,y_n))) \\ &\le \varphi\left(\frac{M(x_{n-1},x_n)+M(y_{n-1},y_n)}{2}\right) - \psi\left(\frac{M(x_{n-1},x_n)+M(y_{n-1},y_n)}{2}\right) \\ &\le \varphi\left(\frac{(M(x_{n-1},x_n)+M(y_{n-1},y_n))}{2}\right) \quad (2)\end{aligned}$$

$$M(x_{n-1},x_n) = \{p(x_{n-1},x_n),p(x_{n-1},x_n),p(x_n,x_{n+1}), \frac{1}{2}[p(x_{n-1},x_{n+1})+p(x_n,x_n)]\}$$

If $x_n = x_{n+1}$, then $x_n = f(x_n,y_n)$
And $y_n = y_{n+1}$,then $y_n = f(y_n,x_n)$

then F has a coupled fixed point. Therefore we assume that $x_n \neq x_{n+1}$ and $y_n \neq y_{n+1}$ for all n ≥ 0. Then $p(x_n, x_{n+1}) \neq 0$ and $p(y_n, y_{n+1}) \neq 0$ let if possible for some n.

$p(x_{n-1}, x_n) < p(x_n, x_{n+1})$ (*) Since $p(x_{n-1}, x_{n+1}) \leq p(x_{n-1}, x_n) + p(x_n, x_{n+1}) - p(x_n, x_n)$

$\frac{1}{2}[p(x_{n-1}, x_{n+1}) + p(x_n, x_n)] \leq p(x_n, x_{n+1})$

(**)

Similarly $\frac{1}{2}[p(y_{n-1}, y_{n+1}) + p(y_n, y_n)] \leq p(y_{n-1}, y_n)$

Now, from (*), (**), we have

$M(x_{n-1}, x_n) = p(x_n, x_{n+1})$

Similarly $M(y_{n-1}, y_n) = p(y_n, y_{n+1})$ from (2)and above inequality

$$\varphi(p(x_n, x_{n+1})) \leq \varphi\left(\frac{p(x_n, x_{n+1}) + p(y_{n+1}, y_n)}{2}\right), \tag{3}$$

And

$$\varphi(p(y_n, y_{n+1})) \leq \varphi\left(\frac{p(y_n, y_{n+1}) + p(x_{n+1}, x_n)}{2}\right). \tag{4}$$

By adding (3), (4), and use $\frac{1}{2}\varphi(t) \leq \varphi(\frac{t}{2})$, we have

$$\begin{aligned} \varphi(p(x_n, x_{n+1})) + \varphi(p(y_n, y_{n+1})) &\leq 2\varphi\left(\frac{p(x_n, x_{n+1}) + p(y_{n+1}, y_n)}{2}\right) \\ \varphi(p(x_n, x_{n+1})) + \varphi(p(y_n, y_{n+1})) &\leq \varphi(p(x_n, x_{n+1}) + p(y_{n+1}, y_n)) \end{aligned}$$

This is contradiction with $\varphi(t+s) \leq \varphi(t) + \varphi(s)$.

Then, $P(x_n, x_{n+1}) \leq p(x_{n-1}, x_n)$ and $P(y_n, y_{n+1}) \leq p(y_{n-1}, y_n)$

Then $M(x_{n-1}, x_n) = p(x_{n-1}, x_n) \quad and \quad M(y_{n-1}, y_n) = p(y_{n-1}, y_n)$ (5)

Since φ is non increasing , from (3) and (4) by (5).

We have
$$p(x_n, x_{n+1}) \leq \left(\frac{p(x_{n-1}, x_n) + p(y_{n-1}. y_n)}{2}\right), \tag{6}$$

$$p(y_n, y_{n+1}) \leq \left(\frac{p(y_{n-1}, y_n) + p(x_{n-1}, x_n)}{2}\right). \tag{7}$$

By adding (6), (7), we have $p(x_n + x_{n+1}) + p(y_n, y_{n+1}) \leq p(y_{n-1}, y_n) + p(x_{n-1}, x_n)$ set tn = p(xn-1, xn)+p(yn, yn+1) , then the sequence tn is non-increasing and bounded below, therefor there is some t ≥ 0 such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty}[p(x_n, x_{n+1}) + p(y_n, y_{n+1})] = t$ (8)

Now ,we will show that t = 0 Assume that t > 0 $\varphi\left(\frac{p(x_n,x_{n+1})+p(y_n,y_{n+1})}{2}\right) \le \varphi(max\{p(x_n,x_{n+1}),p(y_n,y_{n+1})\})$
$\doteq max\{\varphi(p(x_n,x_{n+1})),\varphi(p(y_n,y_{n+1}))\} \le \varphi\left(\frac{p(x_{n-1},x_n)+p((y_{n-1},y_n)}{2}\right)-\psi\left(\frac{p(x_{n-1},x_n)+p((y_{n-1},y_n)}{2}\right).$

Then , taking the limit as $n\to\infty$ and using (2.8) and having in mind that $\lim_{y\to r}\psi(r)>0$ for all r > 0 and φ is continuous, we have

$$\varphi(\frac{t}{2}) = \lim_{n\to\infty}\varphi\left(\frac{t_n}{2}\right) \le \lim_{n\to\infty}\left[\varphi\left(\frac{t_{n-1}}{2}\right)-\psi\left(\frac{t_{n-1}}{2}\right)\right]$$

$$= \varphi\left(\frac{t}{2}\right)-\lim_{t_{n-1}\to t}\psi\left(\frac{t_{n-1}}{2}\right)<\varphi\left(\frac{t}{2}\right),$$ This is contradiction with $\lim_{t\to r}\psi(t)>0,\forall r>0$

then t = 0, Denote $t_n^s = p^s(x_n,x_{n+1})+p^s(y_n,y_{n+1}),\forall n\in N$ from the definition of ps,we get

$$t_{n^s} = p^s(x_n,x_{n+1})-p^s(y_n,y_{n+1}) = 2p(x_n,x_{n+1})-p(x_n,x_{n+})-p(x_{n+1},x_{n+1})+2p(y_n,y_{n+1})$$
$$-p(y_n,y_n)-p(y_{n+1},y_{n+1}) = 2t_n-[p(x_n,x_{n+})+p(x_{n+1},x_{n+1})+p(y_n,y_n)$$
$$+p(y_{n+1},y_{n+1})] \le 2t_n$$ By taking Limit at $n\to\infty$, from (8), using t=0 $0\le\lim_{n\to\infty}t_n^s\le\lim_{n\to\infty}2t_n\le 2\lim_{n\to\infty}t_n$ Then $\lim_{n\to\infty}t_n^s=0$ (9)

Now we prove that xn and yn are cauchy sequences in the partial metric space (X, p), from lemma 1 it is sufficient to prove that xn and yn are cauchy sequences in the metric space (X, ps). suppose to the contrary. So at least one of xn and yn is not a cauchy sequences in (X, ps) . then there exist $\varepsilon>0$ and sequences of natural number (m(k)) and (l(k)) such that for every natural number k $m(k)>l(k)\ge k$, and $r_k^s = p^s(x_{l(k)},x_{m(k)})+p^s(y_{l(k)},y_{m(k)})\ge\varepsilon$

(10)

Now corresponding to l(k) we choose m(k) to be the smallest for which (10) holds.

So $p^s(x_{l(k)},x_{m(k)-1})+p^s(y_{l(k)},y_{m(k)-1})<\varepsilon.$ from (1)

using triangle inequality, we get $\varepsilon\le r_k^s \le p^s(x_{l(k)},x_{m(k)-1})+p^s(x_{m(k)-1},x_{m(k)})$
$+p^s(y_{l(k)},y_{m(k)-1})+p^s(y_{m(k)-1},y_{m(k)}) < \varepsilon+t^s_{m(k)-1}$

Letting $k\to\infty$ and using (9) $\lim_{k\to\infty}r_k^s<\varepsilon+0$, then $\lim_{k\to\infty}r_k^s=\varepsilon$ (11)

On the other hand , let $r_k = p(x_{l(k)},x_{m(k)})+p(y_{l(k)},y_{m(k)})$

By definition of r_k^s we get $r_k^s = p^s(x_{l(k)},x_{m(k)})+p^s(y_{l(k)},y_{m(k)})$ in view of property of (p2) and (8) we get $\lim_{k\to\infty}p(x_{l(k)},x_{l(k)})=\lim_{k\to\infty}p(x_{m(k)},x_{m(k)})+\lim_{k\to\infty}p(y_{l(k)},y_{l(k)})$ $+\lim_{k\to\infty}p(y_{m(k)},y_{m(k)})$ Therefore, letting $k\to\infty$ and using (11), we get $\lim_{k\to\infty}r_k^s=\varepsilon=\lim_{k\to\infty}2r_k=2\lim_{k\to\infty}r_k$, then $\lim_{k\to\infty}r_k=\frac{\varepsilon}{2}$ (12)

since (xn) is a non-decreasing seguence and (yn) is a non-increasing seguence by inequality (1), we have $\varphi(p(x_{l(k)+1},x_{m(k)+1})) = \varphi(p(f(x_{l(k)},y_{l(k)}),p(f(x_{m(k)},y_{m(k)})))$

$$\leq \varphi\left(\frac{M(x_{l(k)},x_{m(k)})+M(y_{l(k)},y_{l(k)})}{2}\right) - \psi\left(\frac{M(x_{l(k)},x_{m(k)})+M(y_{l(k)},y_{l(k)})}{2}\right)$$

Since $M(x_{l(k)},x_{m(k)})=p(x_{l(k)},x_{m(k)})$, and $M(y_{l(k)},y_{m(k)})=p(y_{l(k)},y_{m(k)})$, then

$$\varphi(p(x_{l(k)+1},x_{m(k)+1})) \leq \varphi\left(\frac{r_k}{2}\right)-\psi\left(\frac{r_k}{2}\right)$$

from the tow inequalities , we get using the properties of φ

$$\varphi\left(\frac{p(x_{l(k)+1},x_{m(k)+1})+p(y_{l(k)+1},y_{m(k)+1})}{2}\right) \leq \varphi(max\{p(x_{l(k)+1},x_{m(k)+1}),p(y_{l(k)+1},y_{m(k)+1})\})$$

$\leq \varphi\left(\frac{r_k}{2}\right)-\psi\left(\frac{r_k}{2}\right)$ letting $k \to \infty$ and using properties of φ and ψ together with (12),we have

$\varphi\left(\frac{\varepsilon}{4}\right) \leq \varphi\left(\frac{\varepsilon}{4}\right) - \lim_{k\to\infty}\psi\left(\frac{r_k}{2}\right) = \varphi\left(\frac{\varepsilon}{4}\right) - \lim_{t\to\frac{\varepsilon}{4}}\psi(t) < \varphi\left(\frac{\varepsilon}{4}\right)$ This is contradiction. Therefore xn and yn are cauchy sequences in the metric space (X, ps), since (X, p) is complete, from lemma 1, (X, ps) is complete metric space, then there are $x,y \in X$ such that $\lim_{k\to\infty} p^s(x_n,x) = \lim_{k\to\infty} p^s(y_n,y)=0$

Therfor from (8), using lemma 1 and (p2) $\lim_{n\to\infty}[p(x_n,x)+p(y_n,y)]=0$, then $\lim_{n\to\infty}p(x_n,x)=0$ and $\lim_{n\to\infty}p(y_n,y)=0$ Since $p(x_n,x_n) \leq p(x_n,x_{n+1})$, taking $n\to\infty$ $\lim_{n\to\infty}p(x_n,x_n) \leq \lim_{n\to\infty}p(x_n,x_{n+1})$,
then

$$p(x,x)=\lim_{n\to\infty}p(x_n,x)=\lim_{n\to\infty}p(x_n,x_n)=0$$

Similarly $p(y,y)=\lim_{n\to\infty}p(y_n,y)=\lim_{n\to\infty}p(y_n,y_n)=0$ We shall show that $x=f(x,y)$ and $y=f(y,x)$ (a)Assume that F is continuous on X . in particular ,F is continuous at (x, y),hence for any $\varepsilon>0$, , there exist $\delta>0>$ such that if $(u,v)\in X\times X$ verifying $v((x,y),(u,v))<v((x,y),(x,y))+\delta$

meaning that $p(x,u)+p(y,v)<p(x,x)+p(y,y)+\delta=\delta$

Then we have, $p(F(x,y),F(u,v))<p(F(x,y),F((x,y))+\frac{\varepsilon}{2}$ (13)

Since $\lim_{n\to\infty}p(x_n,x)=\lim_{n\to\infty}p(y_n,y)=0, \quad for \quad \alpha=min(\frac{\delta}{2},\frac{\varepsilon}{2})>0$ there exist n0,m0 $\in$ N such that ,for n >n0 , m> m0, then $p(x_n,x)<\alpha \quad and \quad p(y_n,y)<\alpha$ (14)

Then for $n\in N$, $n\geq max(n_0,m_0)$. we have $p(x_n,x)+p(y_n,y)<2\alpha<\alpha$, so we get $p(F(x,y),x) \leq p(F(x,y),x_{n+1})+p(x_{n+1},x) = p(F(x,y),F(x_n,y_n))+p(x_{n+1},x)$

$$< p(F(x,y),F(x,y))+\frac{\varepsilon}{2}+\alpha$$

From (13) and (14) On the other hand $p(x,x)-p(y,y)=0$ in (1) We get $\varphi(p(F(x,y),F(x,y)))\le\varphi\left(\frac{M(x,x)+M(y,y)}{2}\right)$ $-\psi\left(\frac{M(x,x)+M(y,y)}{2}\right)$. By using (5) $M(x,x)=p(x,x)$ and $M(y,y)=p(y,y)$ Then $\varphi(p(F(x,y),F(x,y)))\le\varphi\left(\frac{p(x,x)+p(y,y)}{2}\right)-\psi\left(\frac{p(x,x)+p(y,y)}{2}\right)$ $\le\varphi(0)-\psi(0)=-\psi(0)\le 0$ Which implies that $p(F(x,y),F(x,y))=0$, so for any $\varepsilon>0$, then $p(F(x,y),x)<0+\varepsilon$, this implies that $F(x,y)=y$, and we can show that $F(y,x)=y$. (b)Assume that X satisfies the two conditions given by (1) and (2). Since xn is a non decreasing sequence and $x_n\to x$ and as yn is a non -increasing sequence and $y_n\to y$ hence we have xn ≤ x and yn ≥ y for all n, by the condition (p4) , we have $p(x,F(x,y))\le p(x,x_{n+1})+p(x_{n+1},F(x,y))=p(x,x_{n+1})+p(F(x_n,y_n),F(x,y))$.

Therfore $\varphi(p(x,F(x,y)))\le\varphi(p(x,x_{n+1}))+\varphi(p(F(x_n,y_n),F(x,y)))\le\varphi(p(x,x_{n+1}))+\varphi\left(\frac{M(x_n,x)+M(y_n,y)}{2}\right)$ $-\psi\left(\frac{M(x_n,x)+M(y_n,y)}{2}\right)$ Taking the limit as $n\to\infty$, using $\lim_{n\to\infty}p(x_n,x)=\lim_{n\to\infty}p(y_n,y)=0$ And the properties of φ and ψ, we have $\varphi(p(x,F(x,y)))=0$,thus $p(x,F(x,y))=0$ Hence $x=F(x,y)$, similarly , one can show that $y=F(y,x)$.

Corollary 2.1. Let (X,≤) be partial ordered set and suppose there is a partial metric P on X such that (X, p) is a complete partial metric space. Let $F:X\times X\to X$ be a mapping having the mixed monotone on X.Assume that there exist $\varphi\in\ \Phi\ and\ \psi\in\Psi$ such that $\varphi(p(F(x,y),F(u,v)))\le\frac{1}{2}(M(x,u)+M(y,v))$ $-\psi\left(\frac{M(x,u)+M(y,v)}{2}\right)$ (15)

Since

$M(x,u)=max\{p(x,u),p(x,F(x,y)),p(u,F(u,v)),\frac{1}{2}[p(u,F(x,y))+p(x,F(u,v))]\}$

and $M(y,v)=max\{p(y,v),p(y,F(y,x)),p(v,F(v,u)),\frac{1}{2}[p(v,F(y,x))+p(y,F(v,u))]\}$

for all $x,y,u,v\in X$ with x ≥ u and y ≤v. suppose either F is continuous or X has the following properties:
(1) if a non-decreasing $x_n\to x$,then xn ≤ x for all n. (2) if a non-increasing $x_n\to x$,then xn ≥ x for all n. if there exist x0, y0 ∈ X such that $x_0\le f(x_0,y_0)$ and $y\ge f(y_0,x_0)$, then There $x,y\in X$ such that $x=f(x,y)$ And $y=F(y,x)$. That is F has a coupled fixed point. Furthermore, $p(x,x)=p(y,y)=0$

Proof. by using $\frac{1}{2}\varphi(t)\le\varphi(\frac{t}{2})$ in theorem

2.1. **Corollary 2.2.** Let (X,≤) be partial ordered set and suppose there is a partial metric P on X such that (X, p) is a complete partial metric space. Let $F:X\times X\to X$ be a mapping having the mixed monotone property on X. Assume that there exists areal number

$k \in [0,1)$ such that $p(F(x,y),F(u,v)) \leq \frac{k}{2}(M(x,u)+M(y,v))$ for all $x,y,u,v \in X$ with x ≥ u and y ≤v. suppose either F is continuous or X has the following properties: (1)if a non-decreasing $x_n \to x$,then xn ≤ x for all n. (2)if a non-increasing $x_n \to x$,then xn ≥ x for all n. if there exist x0, y0 $\in$ X such that $x_0 \leq f(x_0,y_0)$ and $y \geq f(y_0,x_0)$, then There $x,y \in X$ such that $x = f(x,y)$ And $y = F(y,x)$. That is F has a coupled fixed point. Furtharmore, $p(x,x) = p(y,y) = 0$

Proof. We take $\psi(t) = \frac{1-k}{2}t$ in corollary 2.1 Now we shall prove the uniqueness of a coupled fixed point .Note that if (X,≤) is a partially ordered set . then we endow the product $X \times X$ withe the following partial order

For $(x,y),(u,v) \in X \times X$., $(x,y) \leq (u,v) \Leftrightarrow x \leq u, y \geq v.$

Theorem 2.2. In addition to hypotheses of Theorem 2.1., suppose $(x,y),(z,t) \in X \times X$, there exists a (u,v)in $X \times X$ that is comparable to (x, y) and (z, t). Then F has a unique coupled fixed point.

Proof. From theorem 2.1, the set of coupled fixed point of F is non-empty .suppose (x, y) and (z, t) are coupled fixed point of F,that is, x = F(x, y), y = F(y, x), z = F(t, z). We shall show that x = z and y = t. By assumption , there exists $(u,v) \in X \times X$ that is comparable to (x, y) and (z, t). We define sequences {un}, {vn} as follows $u_0 = u$ $,v_0 = v$, $u_{n+1} = F(u_n,v_n)$ and $v_{n+1} = F(v_n,u_n)$ $\forall n.$ Since (u,v) is comparable with (x, y), we may assume that (u0,v0) = (u,v) ≤ (x, y). By using the mathematical induction it is easy to prove that $(u_n,v_n) \leq (x,y) \leq \forall n \in N.$
From (1) we have

$$\varphi(p(x,u_{n+1})) = \varphi(p(F(x,y),F(u_n,v_n))) \leq \varphi\left(\frac{M(x,u_n)+M(y,v_n)}{2}\right) - \psi\left(\frac{M(x,u_n)+M(y,v_n)}{2}\right)$$

$$\leq \varphi\left(\frac{M(x,u_n)+M(y,v_n)}{2}\right)$$

since (u,v) in $X \times X$ that is comparable to (x, y), then from (5) in theorem 2.1 we have $M(x,u_n) = p(x,u_n)$,and $M(y,v_n) = p(y,v_n)$ Then

$$\varphi(p(x,u_{n+1})) \leq \varphi\left(\frac{p(x,u_n)+p(y,v_n)}{2}\right) \tag{16}$$

And

$$\varphi(p(y,v_{n+1})) \leq \varphi\left(\frac{p(y,v_n)+p(x,u_n)}{2}\right) \tag{17}$$

Since φ is non-decreasing, from the above inequalities, we have

$$(p(x,u_{n+1})) \leq \frac{p(x,u_n)+p(y,v_n)}{2} \tag{18}$$

$$(p(y,v_{n+1})) \leq \frac{p(y,v_n)+p(x,u_n)}{2} \tag{19}$$

Adding (18),(19), we get $p(x,u_{n+1})+p(y,v_{n+1})\leq p(x,u_n)+p(y,v_n)$ that is, the sequence {p(x,un)+ p(y,vn)} is a non-increasing. Therefor there exist $\alpha\geq 0$ such that $\lim_{n\to\infty}[p(x,u_n)+p(y,v_n)] = \alpha.$ Now, we shall show that $\alpha=0$. Suppose to the contrary.

By (16),(17),

we have $\varphi\left(\frac{p(x,u_{n+1})+p(y,v_{n+1})}{2}\right) \leq \varphi max\{(p(x,u_{n+1}),p(y,v_{n+1}))\} \leq \max\{\varphi(p(x,u_{n+1})),\varphi(p(y,v_{n+1}))\}$

$$\leq \varphi\left(\frac{M(x,u_n)+M(y,v_n)}{2}\right)-\psi\left(\frac{M(x,u_n)+M(y,v_n)}{2}\right) \leq \varphi\left(\frac{p(x,u_n)+p(y,v_n)}{2}\right)$$

Letting $n\to\infty$, we have

$$\varphi\left(\frac{\alpha}{2}\right) \leq \varphi\left(\frac{\alpha}{2}\right)-\lim_{n\to\infty}\psi\left(\frac{p(x,u_n)+p(y,v_n)}{2}\right) < \varphi\left(\frac{\alpha}{2}\right)$$

A contradiction thus $\alpha=0$, that is $\lim_{n\to\infty}[p(x,u_n)+p(y,v_n)]=0$

It follows that $\lim_{n\to\infty}p(x,u_n)=\lim_{n\to\infty}p(y,v_n)=0$ similarly, one can show that $\lim_{n\to\infty}p(z,u_n)=\lim_{n\to\infty}p(t,v_n)=0$

since p(x, z) ≤ p(x,un)+ p(un, z) and p(y, t) · p(y,vn)+ p(vn, t), etting $n\to+\infty$, we obtain p(x, z) = p(y, t) = 0. so x = z and y = t.

Theorem 2.3. In addition to hypotheses of theorem 2.1, if x0 and y0 are comparable, then x = F(x, y) = F(y, x) = y where (x, y) a coupled fixed point F.

Proof. Following the proof of theorem 1.2. ,F has a coupled fixed point (x, y).We only have to show that x = y. since x0 and y0 are comparable,we may assume that x0 ≥ y0 .By using the mathematical induction ,one can show that xn ≥ yn for any $n\in N$. Not that , by (p4)

$$p(x,y) \leq p(x,x_{n+1})+p(x_{n+1},y_{n+1})+p(y_{n+1},y)$$

$= p(x,x_{n+1})+p(y_{n+1},y)+p(F(x_n,y_n),F(y_n,x_n)).$ Therefore, using the condition (p3),(1) and a property of φ

$$\varphi(p(x,y)) \leq \varphi(p(x,x_{n+1})+p(y_{n+1},y))+\varphi(p(F(x_n,y_n),F(y_n,x_n)))$$

$$\leq \varphi(p(x,x_{n+1})+p(y_{n+1},y))+\varphi(M(x_n,y_n))-\psi(M(x_n,y_n)) \leq \varphi(p(x,x_{n+1})+p(y_{n+1},y)) +\varphi(p(x_n,y_n))-\psi(p(x_n,y_n)) \tag{20}$$

From

$\lim_{n\to\infty}p(x_n,x)=\lim_{n\to\infty}p(x_n,x)=0$ We gave $\lim_{n\to\infty}p(x_n,y_n)=p(x,y).$ Assume that p(x, y) ≠ 0.Letting $n\to\infty$ in (20) we get

$$\begin{aligned}\varphi(p(x,y)) &\leq \varphi(0)+\varphi(p(x,y)))-\lim_{n\to\infty}\psi(p(x_n,y_n)) \\ &= \varphi(p(x,y)))-\lim_{p(x_n,y_n)\to p(x,y)}\psi(p(x_n,y_n)),\end{aligned}$$

That is $\lim_{p(x_n,y_n)\to p(x,y)}\psi(p(x_n,y_n))\leq 0,$ a contradiction. Thus, p(x, y) = 0, so x= y.

Corollary 2.3. Let (X,≤) be partial ordered set and suppose there is a partial metric P on X such that (X, p) is a complete partial metric space. Let $F: X\times X\to X$ be a mapping having the mixed monotone on X.Assume that there exist $\varphi\in\ \Phi\ and\ \psi\in\Psi$ such that

$$\varphi(p(F(x,y),F(u,v)))\leq\frac{1}{2}\varphi(M(x,u)+M(y,v))$$

$$-\psi\left(\frac{M(x,u)+M(y,v)}{2}\right)$$

for all $x,y,u,v\in X$ with x ≥ u and y ≤v. suppose either F is continuous or X has the following properties:

(1) if a non-decreasing $x_n\to x$,then xn ≤ x for all n.
(2) if a non-increasing $x_n\to x$,then xn ≥ x for all n.

if there exist x0, y0 ∈ X such that $x_0\leq f(x_0,y_0)$ and $y\geq f(y_0,x_0)$, then There $x,y\in X$ such that $x=f(x,y)$ And $y=F(y,x)$. That is F has a coupled fixed point.

Furtharmore, $p(x,x)=p(y,y)=0$

Proof. Follows from theorem 2.1.

Corollary 2.4. In addition to hypotheses of corollary 2.1, suppose that for every $(x,t),(z,t)\in X\times X$, there exist a (u, v) in $X\times X$ that is comparable to (x, y) and (z, t) ,then F has a unique coupled fixed point.

Proof. Follows from Theorem 2.2.

Corollary 2.5. In addition to hypotheses of Theorem 2.1, if x0 and y0 are comparable, then x = F(x, y) = F(y, x) = y where (x, y) is a coupled fixed point of F. **Proof**. Follows from Theorem (2.3) and (2.1) **Theorem 2.4 .** Let (X,≤) be partial ordered set and suppose there is a partial metric P on X such that (X, p) is a complete partial metric space. Let $F: X\times X\to X$ be a mapping having the mixed monotone property on X.Then, the following are equivalent: (1)There exist $\varphi,\psi\in\Phi$ such that for any $x,y,u,v\in X$ with x¸ u and y-v, we have

$$\varphi(p(F(x,y),F(u,v)))\leq\varphi\left(\frac{M(x,u)+M(y,v)}{2}\right)-\psi\left(\frac{M(x,u)+M(y,v)}{2}\right). \qquad (21)$$

(2) there exist $\alpha\in[0,1)$ and $\varphi\in\Phi$ such that for any x, y,u,v 2 X with x ¸ u and y · v,we have

$$\varphi(p(F(x,y),F(u,v))) \leq \alpha\varphi\left(\frac{M(x,u)+M(y,v)}{2}\right) \tag{22}$$

(3)there exist a continuous non-decreasing function $\varphi:[0,\infty+)\to[0,\infty+)$ such that $\varphi(t)<t$ for all t > 0 and for any $x,y,u,v\in X$ with y≤ v,we have $p(F(x,y),F(u,v)) \leq \varphi\left(\frac{M(x,u)+M(y,v)}{2}\right)$ (23)

Proof

$$D=\{\frac{(p(x,u)+p(y,v))}{2},p(F(x,y),F(u,v)):x,y,u,v\in X,x\geq u \quad and \quad y\leq v\}$$

Then, the proof follows from (i),(vi) to (vii)off lemma1 of [9]. From Theorem (2.1) ,we have the following remark.**Remark 2.1.** (a)Let φ' and ϕ be as in Theorem 2.4. part (1) and replace inequality (20),Then theorem (2.1),(2.3) are still valued. (b) Let φ be as Theorem 2.4. part (2) and replace inequality (21).Then ,Theorem (2.1),(2.3) are still valued. (c) Let ϕ be as Theorem 2.3. part (3) and replace inequality (22).Then, Theorem (2.1),(2.3) are still valued. Now, we introduce an example to support our results.

Example 2.1

Let X = [0,1] and $p(x,y)=max\{x,y\}$, $F:X\times X\to X\times X$ defined by $F(x,y)=\frac{3}{8}(x,y)$

Let $\varphi(t)=t,\psi(t)=\frac{t}{4},\forall x,y,u,v\in$ Then we have $p(F(x,y),F(u,v))\leq\frac{3}{8}(p(x,u),p(y,v))$

Since $\varphi(p(F(x,y),F(u,v))) = p(F(x,y),F(u,v))=max\{F(x,y),F(u,v)\}$

Then from (1) $\varphi(p(F(x,y),F(u,v)))\leq\varphi\left(\frac{M(x,u)+M(y,v)}{2}\right)-\psi\left(\frac{M(x,u)+M(y,v)}{2}\right)$

Since $M(x,u)=max\{p(x,u),p(x,F(x,y)),p(u,F(u,v)),\frac{1}{2}[p(u,F(x,y))+p(x,F(u,v))]\} = max\{x,x,u,\frac{x+x}{2}\}=x$

Then $M(y,v) = max\{v,y,v,\frac{y+v}{2}\}=v$

Then we have $\left(\frac{M(x,u)+M(y,v)}{2}\right)-\left(\frac{M(x,u)+M(y,v)}{2}\right)=\frac{x+v}{2}$

then $\varphi\left(\frac{M(x,u)+M(y,v)}{2}\right)-\psi\left(\frac{M(x,u)+M(y,v)}{2}\right) = \frac{x+v}{2}-\frac{x+v}{8}$

$$=\frac{3(x+v)}{8}$$
$$\geq \frac{3}{8}x\geq\frac{3}{8}xy$$
$$=\varphi(p(F(x,y),F(u,v)))$$

So F has a unique fixed point (0,0) in X

References

1. M.Abbas, M.Ali Khan and S.Radenovic, Common coupled fixed point theorems in con metric spaces for w-compatible mappings, Appl. Math.Comput.217 (2010), no.1, 195202.
2. T. Abdeljawad, E. Karapinar, K. Tas, Existence and uniquness of acmmon fixed point partial metric spaces, Appl. Math. Lett.24 (2011)1900-1904.
3. H. Aydi, Erdal Karapinar and Wasfi Shatanawi, Coupled fixed point results for ($\tilde{A}$,′)-Weakly contractive condition in ordered partial metric spaces,Appl. Math.Comput,(2011),4449-4460.
4. T.Gnana Bhaskar,V. Lakshmikantham, Fixed point theorems in partailly ordered metric spaces and applications,
5. Nonlinear Anal. 65(2006), no1379-1393.
6. T.G. Bhashkar, V. Lakshmikantham, Fixed point throrems in partially ordered cone metric space and applications, Nonlinear Anal. 65 (7) (2006) 825-832.
7. S.S.Chang, Ma, Y.H., Coupled fixed point of mixed monotone condensing operators and existence theorem of the solution for a class of functions arising in dynamic programming ,J.Math. Anal.Appl.160 (1991),468-479.
8. B.S.Choudhury and P.Maity, Coupled fixed point results in generalized metric spaces,
9. Math.Comput.Modelling, 54,(2011),no.1-2,73-79.
10. D. Ilic, V. Pavlovic, V. Rakocecic, Some new extensinos of Banach's contraction principle to partial metric space, Appl, Math, Lett, 24(8)(2011)1326-1330.
11. J. Jachymski, Equivalent conditions for generalized contraction on ordered metric spaces, Appl. Math. Lett.24 (8)(2011).
12. E. Karapinar, Generalizations of Caristi Kirk's Theorem on partial metric space,Fixed point Theory Appl.2011(1)(2011)4.
13. V. Lakshmikantham and L. Ciric, Coupled fixed point theorems for nonlinear contraction in partially ordered metric spaces Nonlinear Anal.70 (2009), 4341-4349.
14. N. V.Luong, N. Xuan Thuan, Coupled in fixed point theorems in partailly ordered metric spaces and application, Nonlinear Anal. 74 (2011) 983-992.
15. S.G. Matthews: Partial metric topology. Research Report 212. Dept. of Computer Science. University of Warwick, 1992.
16. S.G. Matthews, prtial metric topology in: proc.8thsummer Conference on General Topology and Application, in: Ann. New York Acad. Sci.728 (1994), p. 183-197.
17. H.K. Nashine and I. Altun, fixed pointtheorems for generalized weakly contractive condition in ordered metric spaces, Fixed Point Theory Appl., In Press.
18. X.T.Nguyen, Coupled fixed point in partially ordered metric spaces and application, Nonlinear
19. Anal., 74(2011),983992.

20. F.Sabetghadam, H.P. Masiha and A.H. Sanatpour, Some coupled fixed point theorems in cone metric spaces, Fixed Point Theory Appl., 2009, Art.ID 125426, 8 PP.
21. S.Sedghi, I. Altun and N.Shobe, Coupled fixed point theorems for contraction in fuzzy metric spaces, Nonlinear Anal., 72(2010), no. 3-4, 12981304.
22. J. E. Stoy, Denotational Semantics: The Scott-Strachey Approach to Programming Language Theory, MIT Press Cambridge, (1981).